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ANALYSIS OF THE STABILITY OF A THIN LIQUID FILM ADJACENT TO A HIGH-SPEED GAS STREAM

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SUMMARY

Oblique wave formation in the melt layer of a body entering the atmosphere is examined. The three-dimensional disturbance equations of the liquid including the effects of viscosity are formulated. The disturbance motion of the gas is taken into account neglecting viscosity. The Tollmien-Schlichting type of instability in the liquid is excluded and only those modes are considered for which the phase velocity of the disturbance is greater than the velocity of the basic liquid flow at the gas-liquid interface. Numerical and approximate analytic results are presented for waves in the liquid with supersonic flow over their crests. A comparison with the numerical solutions shows that the approximate analytic solutions are accurate for solving the eigenvalue problem when the Reynolds number of the liquid film is small, that is, for a highly viscous melt layer. The approximate solutions are obtained in closed form. It is demonstrated that the mechanism of liquid wave generation is supersonic wave drag. The relation between the wavelength of the disturbance and the other parameters involved in the eigenvalue problem is determined and is presented for a wide range of these parameters.

INTRODUCTION

The theory of hydrodynamic stability has been employed by several authors in analyzing the formation of waves in the melt layer of a body entering the atmosphere.

Feldman (ref. 1) considered a liquid shear flow with wave crests parallel to the stream direction (streamwise grooves) but neglected the disturbance motion of the gas. Miles (ref. 2) considered a liquid shear flow with wave crests normal to the stream direction, and although he did not include the disturbance motion of the gas in calculating the stability of the liquid film, he did indicate that the disturbance motion of the gas cannot be relegated to a subsidiary role when the phase velocity of the disturbance wave is greater than the velocity of the liquid at the gas-liquid interface. Chang and Russell (ref. 3) considered wave crests normal to the stream direction and they included the disturbance motion of the gas in their analysis. However, the undisturbed liquid configuration considered by them was the classic Kelvin-Helmholtz type, as treated by Lamb (ref. 4), namely, an infinitely deep, initially quiescent liquid suddenly subjected to a disturbance, periodic in time.

The waves considered in this paper are the type observed, for instance, in reference 5. The crests of the waves are oblique to the stream direction (fig. 1); the problem considered is the three-dimensional disturbance of a shear flow of a thin liquid film adjacent to a high-speed gas stream. The disturbance motion of the gas is accounted for in a manner similar to that employed by Chang and Russell (ref. 3).

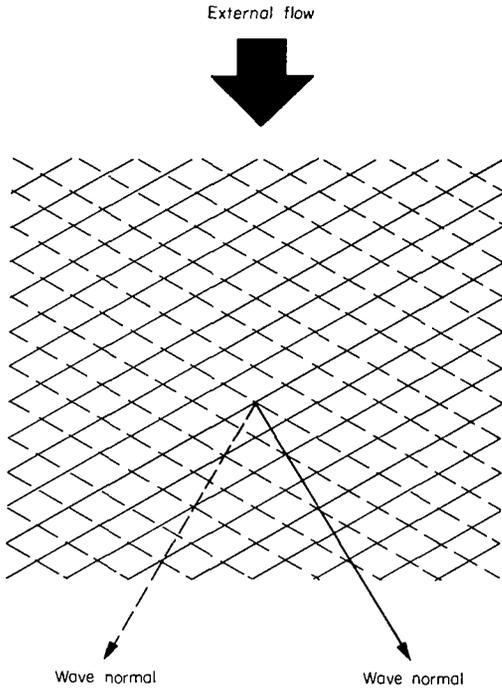


Figure 1.- Two sets of oblique waves.

The problem of oblique (three-dimensional) wave formation cannot, in general, be transformed to a two-dimensional problem by employing Squires' theorem (ref. 6). For example, in the case of a supersonic gas stream, the gas flow over the crests of the waves in the liquid film may be either subsonic or supersonic, depending on the direction of wave propagation. In subsonic flow, the disturbance pressure of the gas is in phase with the interface whereas in supersonic flow it is in phase with the slope of the interface. Clearly, the stability problem is different for the two cases and therefore requires a three-dimensional treatment. The primary interest in the present investigation is the coupling between the disturbance motion of the gas and the stability of the liquid. Specifically excluded are modes of energy transfer as treated by Miles (ref. 2), that

transfer energy from the mean motion of the liquid to its disturbance motion. As Miles pointed out this type of energy transfer occurs in the Tollmien-Schlichting type of instability where the phase velocity of the disturbance wave is less than the velocity of the liquid at the interface. Hence, to study the effect of the disturbances in the gas on the stability of the liquid film, consideration will be given to those possible modes mentioned by Miles (ref. 2) for which the phase velocity of the wave is greater than or equal to the velocity of the liquid at the interface. Of course, considering these "fast" waves in the liquid raises the question of the Tollmien-Schlichting type of instability in the gas, but this question can be ignored since in the majority of practical applications the gas flow is likely to be turbulent.

In the following sections the three-dimensional disturbance equations are formulated for the liquid and the gas, and for the boundary conditions. The equations are then made dimensionless.

A numerical method of solving the eigenvalue problem and an approximate analytic solution are presented. The approximate solutions are compared with the numerical solutions. Then neutral curves resulting from the numerical solution are presented for a selected range of the pertinent parameters. Finally, the approximate solutions are presented in dimensional form for a wide range of parameters for the small Reynolds numbers of the liquid film.

FORMULATION OF DISTURBANCE EQUATIONS

Consider the stratified motion of a thin film of liquid over a solid surface adjacent to a high-speed gas stream. For analyzing the stability of this flow configuration, a perturbation is introduced into the describing equations for the liquid and the gas. The basic liquid flow is considered to be a shear flow, and viscous effects are included in formulating the disturbance equations of the liquid. The effect of viscosity is neglected in the disturbance motion of the gas. The only effect of the gas considered in the stability analysis is the pressure variation of the gas at the disturbed interface.

Gradients in the basic flow of the gas are neglected except at the interface where it is recognized that the shear in the gas balances the shear in the basic flow of the liquid. Away from the interface, the basic properties of the gas flow that influence the pressure disturbance are characterized by parameters regarded as constants throughout the gas boundary layer. The purpose of this investigation is to determine how those parameters affect the stability of the thin liquid film.

Cartesian coordinates are introduced so that the interface coincides with the (x_1, x_2) plane. The film of liquid flows over a solid wall, located at $x_3 = -h$, so that the x_3 axis points away from the wall. Cartesian tensor notation is used. The subscripts α and β range from 1 and 2 (two mutually perpendicular directions in the plane of the interface), and repeated subscripts imply a tensor summation over this range. The basic flow is taken to be in the x_1 direction; however, a three-dimensional disturbance is considered. Furthermore, the wave amplitude is assumed to be sufficiently small that the principle of superposition applies. Therefore, the two families of wave crests shown in figure 1 may be treated independently.

DISTURBANCE EQUATIONS - LIQUID

Consider a three-dimensional disturbance of the basic flow whereby only first-order terms are retained:

$$\left. \begin{aligned} \bar{u}_\alpha(x_1, x_2, x_3, t) &= U_\alpha(x_3) + u_\alpha(x_3) E \\ \bar{u}_3(x_1, x_2, x_3, t) &= u_3(x_3) E \\ \bar{p}(x_1, x_2, x_3, t) &= P(x_3) + p(x_3) E \end{aligned} \right\} \quad (1)$$

where

$$E = \exp i(k_\beta x_\beta - \omega t)$$

The wave numbers k_1, k_2 are taken to be real. The wave normal has the direction numbers

$$(k_1, k_2, 0)$$

The wave length of the disturbance in the direction of the wave normal is given by $2\pi/k$ where $k = \sqrt{k_\alpha^2}$. The constant ω is taken to be complex. The real part of ω is equal to the angular frequency of the disturbance, and the imaginary part is equal to the time amplification factor of the disturbance.

The appropriate form of the disturbance equations is derived by substituting the form of the disturbance introduced above into the equations of motion of a viscous liquid

$$\left. \begin{aligned} \frac{\partial \bar{u}_\alpha}{\partial t} + \bar{u}_\beta \frac{\partial \bar{u}_\alpha}{\partial x_\beta} + \bar{u}_3 \frac{\partial \bar{u}_\alpha}{\partial x_3} &= -\frac{1}{\rho} \frac{\partial \bar{p}}{\partial x_\alpha} + \nu \left(\frac{\partial^2 \bar{u}_\alpha}{\partial x_\beta^2} + \frac{\partial^2 \bar{u}_\alpha}{\partial x_3^2} \right) \\ \frac{\partial \bar{u}_3}{\partial t} + \bar{u}_\beta \frac{\partial \bar{u}_3}{\partial x_\beta} + \bar{u}_3 \frac{\partial \bar{u}_3}{\partial x_3} &= -\frac{1}{\rho} \frac{\partial \bar{p}}{\partial x_3} - g + \nu \left(\frac{\partial^2 \bar{u}_3}{\partial x_\beta^2} + \frac{\partial^2 \bar{u}_3}{\partial x_3^2} \right) \\ \frac{\partial \bar{u}_\beta}{\partial x_\beta} &= 0 \end{aligned} \right\} \quad (2)$$

and retaining first-order disturbance quantities. (Symbols are defined in appendix A.) The basic flow satisfies

$$\left. \begin{aligned} U_1(x_3) &= U_x(1 + x_3/h) \\ U_2(x_3) &= 0 \\ P(x_3) &= P_g - \rho g x_3 \end{aligned} \right\} \quad -h \leq x_3 \leq 0 \quad (3)$$

where P_g is the pressure at the undisturbed interface. It is considered sufficiently general to allow the gravity vector to be normal to the (x_1, x_2) plane. The disturbance equations are as follows:

$$\left. \begin{aligned} i(k_\beta U_\beta - \omega)u_\alpha + u_3 \frac{dU_\alpha}{dx_3} &= -i \frac{k_\alpha}{\rho} p + \nu \left(\frac{d^2}{dx_3^2} u_\alpha - k^2 u_\alpha \right) \\ i(k_\beta U_\beta - \omega)u_3 &= -\frac{1}{\rho} \frac{dp}{dx_3} + \nu \left(\frac{d^2}{dx_3^2} u_3 - k^2 u_3 \right) \\ i(k_\beta U_\beta) + \frac{du_3}{dx_3} &= 0 \end{aligned} \right\} \quad (4)$$

In obtaining the disturbance equations, the fact that the basic flow satisfies equations (3) has been utilized. When u_α ($\alpha = 1, 2$) is eliminated from the foregoing equations the disturbance pressure p is expressed in terms of u_3

$$p = \frac{\rho}{k} \left\{ i \left[u_3 \frac{dU}{dx_3} - \left(U - \frac{\omega}{k} \right) \frac{du_3}{dx_3} \right] + \frac{\nu}{k} \left(\frac{d^3 u_3}{dx_3^3} - k^2 \frac{du_3}{dx_3} \right) \right\} \quad (5)$$

where

$$U \equiv \frac{k_\alpha}{k} U_\alpha$$

is the velocity component of the basic flow in the direction of the wave normal. Eliminating p from the resulting equations yields the usual Orr-Sommerfeld equation for the liquid film

$$\frac{d^4 u_3}{dx_3^4} - 2k^2 \frac{d^2 u_3}{dx_3^2} + k^4 u_3 = \frac{ik}{\nu} \left[\left(U - \frac{\omega}{k} \right) \left(\frac{d^2 u_3}{dx_3^2} - k^2 u_3 \right) - \frac{d^2 U}{dx_3^2} u_3 \right] \quad (6)$$

For a linear profile, $d^2 U / dx_3^2 = 0$.

DISTURBANCE EQUATIONS - GAS

The effect of viscosity is neglected in the disturbance motion of the gas. The following three-dimensional disturbance is considered:

$$\left. \begin{aligned} \bar{u}_\alpha(x_1, x_2, x_3, t) &= U_\alpha + u_\alpha(x_3) E \\ \bar{u}_3(x_1, x_2, x_3, t) &= u_3(x_3) E \\ \bar{p} &= P_g + p(x_3) E \\ \bar{\rho} &= \rho_g + \rho(x_3) E \end{aligned} \right\} \quad (7)$$

where

$$E = \exp i(k_\beta x_\beta - \omega t)$$

and

$$\bar{p} = \bar{\rho} \frac{\bar{a}^2}{\gamma}, \quad \frac{p}{\rho} = \gamma \frac{p_\infty}{\rho_\infty}$$

Since viscosity is neglected in the disturbance motion of the gas, the appropriate form of the disturbance equations may be obtained by substituting the form of the disturbance introduced above into the equations of motion of a compressible, inviscid gas.

$$\left. \begin{aligned} \frac{\partial \bar{u}_\alpha}{\partial t} + \bar{u}_\beta \frac{\partial \bar{u}_\alpha}{\partial x_\beta} + \bar{u}_3 \frac{\partial \bar{u}_\alpha}{\partial x_3} &= - \frac{\bar{a}^2}{\bar{\rho}} \frac{\partial \bar{\rho}}{\partial x_\alpha} \\ \frac{\partial \bar{u}_3}{\partial t} + \bar{u}_\beta \frac{\partial \bar{u}_3}{\partial x_\beta} + \bar{u}_3 \frac{\partial \bar{u}_3}{\partial x_3} &= - \frac{\bar{a}^2}{\bar{\rho}} \frac{\partial \bar{\rho}}{\partial x_3} \\ \frac{\partial \bar{\rho}}{\partial t} + \bar{u}_\beta \frac{\partial \bar{\rho}}{\partial x_\beta} &= -\bar{\rho} \left(\frac{\partial \bar{u}_\beta}{\partial x_\beta} \right) \end{aligned} \right\} \quad (8)$$

and retaining first-order disturbance quantities. The basic flow satisfies

$$\left. \begin{aligned} U_1 &= U_\infty \\ U_2 &= 0 \\ P &= P_g \\ \rho &= \rho_g \end{aligned} \right\} \quad 0 \leq x_3 \leq \infty \quad (9)$$

where all basic flow quantities are assumed constant. The basic flow may be regarded as a suitable mean of an actual boundary-layer flow.

The disturbance equations are as follows:

$$\left. \begin{aligned} i(k_\beta U_\beta - \omega)u_\alpha &= - \frac{a_\infty^2}{\rho_\infty} i k_\alpha \rho \\ i(k_\beta U_\beta - \omega)u_3 &= - \frac{a_\infty^2}{\rho_\infty} \frac{d\rho}{dx_3} \\ i(k_\beta U_\beta - \omega)\rho &= -\rho_\infty \left[i(k_\beta U_\beta) + \frac{du_3}{dx_3} \right] \end{aligned} \right\} \quad (10)$$

Eliminating u_α ($\alpha = 1, 2$) from the foregoing equations and employing the relation

$$p = a_\infty^2 \rho$$

yields an expression for the disturbance pressure in terms of u_3

$$p = \frac{i p_\infty}{k} \frac{(U_g - \omega/k) (du_3/dx_3)}{[(U_g - \omega/k)^2/a_\infty^2] - 1} \quad (11)$$

where

$$U_g \equiv \frac{k_\beta U_\beta}{k}$$

is the velocity component of the basic flow in the direction of the wave normal. When p is eliminated from the resulting equations the disturbance equation for the gas is

$$\frac{d^2 u_3}{dx_3^2} + k^2 \left[\frac{(U_g - \omega/k)^2}{a_\infty^2} - 1 \right] u_3 = 0 \quad (12)$$

Since the real part of ω/k equals the phase velocity of the disturbed interface, the choice of the proper solutions of equation (12) depends on whether the component of the basic flow in the direction of the wave normal, U_g , relative to the phase velocity is greater or less than the speed of sound a_∞ . If it is greater, equation (12) is solved for a disturbance that originates at the interface.

Supersonic solution:

$$u_3 = A \exp - ik \sqrt{\frac{(U_g - \omega/k)^2}{a_\infty^2} - 1} x_3 \quad (13)$$

where A is an arbitrary constant. If the phase velocity is less than a_∞ (subsonic), equation (12) is solved for a disturbance that decays as the distance from the wall is increased in the positive direction of x_3 .

Subsonic solution:

$$u_3 = A \exp - k \sqrt{1 - \frac{(U_g - \omega/k)^2}{a_\infty^2}} x_3 \quad (14)$$

The disturbance pressure in terms of u_3 for the two cases is:

Supersonic

$$p = \frac{\rho_\infty (U_g - \omega/k)}{\sqrt{[(U_g - \omega/k)^2/a_\infty^2] - 1}} u_3 \quad (15)$$

Subsonic

$$p = \frac{i\rho_\infty (U_g - \omega/k)}{\sqrt{1 - [(U_g - \omega/k)^2/a_\infty^2]}} u_3 \quad (16)$$

The kinematic condition at the interface enables one to evaluate the arbitrary constant A in the above expressions for u_3 in terms of the amplitude of the displacement of the interface. Let the equation of the interface be

$$\bar{x}_3(x_1, x_2, t) = \varepsilon \exp i(k_\beta x_\beta - \omega t) \quad (17)$$

The kinematic condition in the gas at the interface is

$$\bar{u}_3 = \frac{\partial \bar{x}_3}{\partial t} + \bar{u}_\beta \frac{\partial \bar{x}_3}{\partial x_\beta} \quad (18)$$

Neglecting second-order disturbance quantities, and evaluating u_3 at $x_3 = 0$ yields

$$u_3 = ik(U_g - \omega/k)\varepsilon \quad (19)$$

This relation enables one to evaluate the disturbance pressure of the gas at the interface $x_3 = 0$ in terms of ε , the amplitude of the displacement. For the two cases:

Supersonic

$$p = \frac{ik\rho_\infty (U_g - \omega/k)^2 \varepsilon}{\sqrt{[(U_g - \omega/k)^2/a_\infty^2] - 1}} \quad (20)$$

Subsonic

$$p = \frac{-k\rho_\infty (U_g - \omega/k)\varepsilon}{\sqrt{1 - [(U_g - \omega/k)^2/a_\infty^2]}} \quad (21)$$

Subsequently these expressions for p will be used in formulating the boundary conditions of the liquid at the interface. For this purpose, it is considered sufficiently accurate to neglect the term ω/k compared to the term U_g in the above expressions. The expressions that will be used subsequently for the gas pressure are:

Supersonic

$$p = \frac{ik\rho_{\infty}U_g^2\varepsilon}{\sqrt{M^2 - 1}} \quad (22)$$

Subsonic

$$p = \frac{-k\rho_{\infty}U_g^2\varepsilon}{\sqrt{1 - M^2}} \quad (23)$$

where $M \equiv U_g/a_{\infty}$. Note that the Mach number so defined is the projection of the actual Mach number of the basic flow in the direction of the wave normal. Hence, the flow may be subsonic across the wave crests when the basic flow is supersonic.

Boundary Conditions

The boundary conditions at the interface are evaluated at the free surface. The equation of the interface is

$$\bar{x}_3(x_1, x_2, t) = \varepsilon \exp i(k_{\beta}x_{\beta} - \omega t) \quad (24)$$

and the kinematic condition in the liquid at the interface is

$$\bar{u}_3 = \frac{\partial \bar{x}_3}{\partial t} + \bar{u}_{\beta} \frac{\partial \bar{x}_3}{\partial x_{\beta}} \quad (25)$$

Neglecting second-order disturbance quantities and evaluating u_3 at $x_3 = 0$ yields

$$u_3 = ik(U_1 - \omega/k)\varepsilon \quad (26)$$

where U_1 is the velocity of the liquid at the undisturbed interface in the direction of the wave normal. The first boundary condition to be imposed at the interface is that the discontinuity of normal stress equal the surface tension. We have for the normal stress

$$-\bar{p} \Big|_1^g + 2\mu \frac{\partial u_3}{\partial x_3} \Big|_1^g = -T \frac{d^2 \bar{x}_3}{dx_{\alpha}^2} \quad (27)$$

where the second-order terms in the expression for the interface curvature have been neglected, and where T denotes the surface tension.

The above expression evaluated at the disturbed interface is:

$$-\left(P + \frac{dP}{dx_3} \varepsilon + p\right)_g + \left(P + \frac{dP}{dx_3} \varepsilon + p\right)_l + \left(2\mu \frac{\partial u_3}{\partial x_3}\right)_g - \left(2\mu \frac{\partial u_3}{\partial x_3}\right)_l = Tk^2 \varepsilon \quad (28)$$

If the effect of viscosity is neglected in the disturbance motion of the gas and the relations satisfied by the basic flow are cancelled,

$$-p_g - \rho g \varepsilon + p - 2\mu \frac{\partial u_3}{\partial x_3} = Tk^2 \varepsilon \quad (29)$$

In the above expression p_g is the only term that involves the gas, and all quantities are to be evaluated at $x_3 = 0$.

Substituting the relations given for p , p_g , and ε by equations (5), (22), and (26), respectively, into equation (29) yields the normal stress condition for a supersonic gas flow

$$\begin{aligned} (U_l - \omega/k) \frac{du_3}{dx_3} - u_3 \frac{dU}{dx_3} + \frac{i\nu}{k} \left(\frac{d^3 u_3}{dx_3^3} - 3k^2 \frac{du_3}{dx_3} \right) \\ = \frac{1}{\rho} \left(Tk^2 + \rho g + \frac{i k \rho_g U_g^2}{\sqrt{M^2 - 1}} \right) \left[\frac{u_3}{(U_l - \omega/k)} \right] \end{aligned} \quad (30)$$

and for a subsonic gas flow from equation (23)

$$\begin{aligned} (U_l - \omega/k) \frac{du_3}{dx_3} - u_3 \frac{dU}{dx_3} + \frac{i\nu}{k} \left(\frac{d^3 u_3}{dx_3^3} - 3k^2 \frac{du_3}{dx_3} \right) \\ = \frac{1}{\rho} \left(Tk^2 + \rho g - \frac{k \rho_g U_g^2}{\sqrt{1 - M^2}} \right) \frac{u_3}{(U_l - \omega/k)} \end{aligned} \quad (31)$$

The second boundary condition to be imposed at the interface is continuity of tangential stress, that is

$$\left. \begin{aligned} \tau_{13}|_g &= \mu \left(\frac{\partial \bar{u}_3}{\partial x_1} + \frac{\partial \bar{u}_1}{\partial x_3} \right) \Big|_l \\ \tau_{23}|_g &= \mu \left(\frac{\partial \bar{u}_3}{\partial x_2} + \frac{\partial \bar{u}_2}{\partial x_3} \right) \Big|_l \end{aligned} \right\} \quad (32)$$

When the effect of viscosity in the disturbance motion of the gas is neglected,

$$\left. \begin{aligned} \mu \left(ik_1 u_3 \frac{d^2 U_1}{dx_3^2} \epsilon + \frac{du_1}{dx_3} \right) &= 0 \\ \mu \left(ik_2 u_3 + \frac{du_2}{dx_3} \right) &= 0 \end{aligned} \right\} \quad (33)$$

Eliminating u_α ($\alpha = 1, 2$) for the foregoing equation and employing (4) yields

$$\frac{d^3 u_3}{dx_3^2} + k^2 u_3 - \frac{k_\beta}{k} \frac{d^2 U_\beta / dx_3^2}{U_1 - \omega/k} = 0 \quad (34)$$

For a linear velocity profile in the liquid, the last term in equation (34) vanishes. In obtaining this boundary condition the following relation satisfied by the basic flow has been utilized

$$\mu \frac{U_1}{h} = \tau \quad (35)$$

where τ is the shear stress of the basic flow of gas exerted on the interface in the direction of the wave normal

$$\tau \equiv \frac{k_\beta \tau_{\beta 3}}{k} \Big|_g$$

Dimensionless Form of the Equations

The disturbance equations are made dimensionless by a suitable choice of reference dimensional quantities. In the present case the depth of the film h is chosen as the reference length, and the velocity of the film at the undisturbed interface in the direction of the wave normal U_1 is chosen as the reference velocity; the remaining reference quantity is the liquid density ρ . For a linear velocity profile in the liquid the disturbance differential equation (6) is transformed to

$$\phi'''' - 2\alpha^2\phi'' + \alpha^4\phi = i\alpha R[(F' - c)(\phi'' - \alpha^2\phi)] \quad (36)$$

where primes denote differentiation with respect to η and where

$$F' = U/U_1 = 1 + \eta: \quad -1 \leq \eta \leq 0$$

$$c = (\omega/k)/U_1$$

$$\alpha = kh$$

$$\phi = (i/kh)(u_3/U_1)$$

$$R = U_1 h/\nu$$

For the supersonic case the boundary conditions at the interface-equations (30) are transformed as follows:

$$(1 - c)\phi' - \phi - \frac{1}{i\alpha R}(\phi'''' - 3\alpha^2\phi') = \left[\frac{1}{F^2} + \frac{\alpha^2}{W^2} + \frac{i\alpha}{c_f R \sqrt{M^2 - 1}} \right] \frac{\phi}{1 - c} \quad (37)$$

and for the subsonic case

$$(1 - c)\phi' - \phi - \frac{1}{i\alpha R}(\phi'''' - 3\alpha^2\phi') = \left(\frac{1}{F^2} + \frac{\alpha^2}{W^2} - \frac{\alpha}{c_f R \sqrt{1 - M^2}} \right) \frac{\phi}{1 - c} \quad (38)$$

where

$$F^2 = \frac{U_1^2}{gh}$$

$$W^2 = \frac{U_1^2 \rho h}{T}$$

$$\frac{1}{c_f R} = \frac{\rho_g U_g^2}{\rho_1 U_1^2}$$

F and W are the Froude and Weber numbers, respectively. The boundary conditions at the solid surface are

$$\eta = -1; \quad \phi = \phi' = 0 \quad (39)$$

The differential equation (36) and equations (37) to (39) for the boundary conditions constitute an eigenvalue problem for determining the eigenvalue c for given values of α , R , F , W , c_f , and M . Another interpretation of the parameter c_f can be examined if we note that since $\tau = \mu(U_1/h)$

$$\tau = c_f \rho_g U_g^2 \quad (40)$$

Numerical Method of Solution of Eigenvalue Problem

The eigenvalue problem formulated in the previous section is solved in the manner employed in reference 7; namely, the parameters α , R , F , W , c_f , and M are fixed. Then values of c are found for which equation (36) has solutions satisfying the boundary conditions (eqs. (37)-(39)).

The eigenvalue c and the corresponding eigenfunctions are obtained by treating equation (36) as a nonlinear equation. A trial solution is obtained by step-by-step numerical integration of the equation starting at $\eta = 0$ with assumed starting values and with an assumed value of c . The boundary conditions (eqs. (39)), are evaluated at the solid surface $\eta = -1$. If the boundary conditions are not satisfied, the starting values and c are adjusted by means of the Newton-Raphson procedure and another trial is made. To carry out the solution of the eigenvalue problem it is convenient to rewrite the fourth-order differential equation (36) as a system of second-order differential equations. For this purpose, the variable $s = \phi'' - \alpha^2 \phi$ which is related to the disturbance vorticity is introduced, and the differential equation (36) and the boundary conditions equations (37)-(39) are written in terms of s as follows:

$$\left. \begin{aligned} \phi'' &= \alpha^2 \phi + s \\ s'' &= \alpha^2 s + i\alpha R(1 + \eta - c)s \end{aligned} \right\} \quad (41)$$

at $\eta = 0$

$$\left. \begin{aligned} s + 2\alpha^2 \phi &= 0 \\ s' - 2\alpha^2 \phi' - i\alpha \left\{ \frac{f}{c-1} \phi - R[(c-1)\phi' + \phi] \right\} &= 0 \end{aligned} \right\} \quad (42)$$

at $\eta = -1$

$$\phi = \phi' = 0 \quad (43)$$

where for the supersonic case

$$f = R \left(\frac{1}{F^2} + \frac{\alpha^2}{W^2} \right) + \frac{i\alpha}{c_f} \frac{1}{\sqrt{M^2 - 1}} \quad (44)$$

and for the subsonic case

$$f = R \left(\frac{1}{F^2} + \frac{\alpha^2}{W^2} \right) - \frac{\alpha}{c_f} \frac{1}{\sqrt{1 - M^2}} \quad (45)$$

To carry out the numerical integration of equations (41) four initial conditions and a value for c must be specified. It is assumed that the other parameters have been specified. Since the differential equation and the boundary conditions are homogeneous, it is permissible to set $\phi = 1$ at $\eta = 0$. This fixes one of the initial conditions. The remaining initial conditions may be obtained from the boundary conditions, provided one of them is specified. Designate as x the value of $\phi'(0)$. The quantities x and c must be adjusted by trial so as to satisfy the boundary conditions at $\eta = -1$. The initial conditions at $\eta = 0$ are given in terms of x and c as follows:

$$\left. \begin{aligned} \phi &= 1 \\ \phi' &= x \\ s &= -2\alpha^2 \\ s' &= 2\alpha^2 x + i\alpha \left\{ \frac{f}{c-1} - R[(c-1)x + 1] \right\} \end{aligned} \right\} \quad (46)$$

To satisfy the boundary conditions at $\eta = -1$ the values $x \equiv \phi'(0)$ and c are adjusted according to the Newton-Raphson procedure. To carry out this procedure it is necessary to know how ϕ and ϕ' vary with x and c at $\eta = -1$. This information can be supplied by formulating the perturbation equations (ref. 7) associated with equations (41). The perturbation equations are obtained by differentiating the terms in equations (41) with respect to x and c . The appropriate initial values for the perturbation differential equations are obtained by differentiating the terms in the boundary conditions (eqs. (46)) with respect to x and c . Integrating the perturbation differential equations and the original differential equations (41) enables one to evaluate at $\eta = -1$ the pertinent quantities: ϕ , ϕ' , ϕ_x , ϕ_x' , ϕ_c , and ϕ_c' . The subscripts x and c denote differentiation with respect to x and c , respectively.

The perturbation equations for x are

$$\left. \begin{aligned} \phi_x'' &= \alpha^2 \phi_x + s_x \\ s_x'' &= \alpha^2 s_x + i\alpha R(F' - c)s_x \end{aligned} \right\} \quad (47)$$

with the initial conditions

$$\left. \begin{aligned} \phi_x &= 0 \\ \phi_x' &= 1 \\ s_x &= 0 \\ s_x' &= 2\alpha^2 - i\alpha R(c-1) \end{aligned} \right\} \quad (48)$$

The perturbation equations for c are

$$\left. \begin{aligned} \phi_c'' &= \alpha^2 \phi_c + s_c \\ s_c'' &= \alpha^2 s_c + i\alpha R(F' - c)s_c - i\alpha R s \end{aligned} \right\} \quad (49)$$

with the initial conditions

$$\left. \begin{aligned} \phi_c &= 0 \\ \phi_c' &= 0 \\ s_c &= 0 \\ s_c' &= -i\alpha \left[\frac{f}{(c-1)^2} + Rx \right] \end{aligned} \right\} \quad (50)$$

For values of x and c that yield at $\eta = -1$ approximate zeros of ϕ and ϕ' , a better approximation ($x + \Delta x$ and $c + \Delta c$) is obtained by calculating the corrections Δx and Δc from the Newton-Raphson equations

$$\left. \begin{aligned} 0 &= \phi + \phi_x \Delta x + \phi_c \Delta c \\ 0 &= \phi' + \phi_x' \Delta x + \phi_c' \Delta c \end{aligned} \right\} \quad (51)$$

For given values of the parameters, the procedure outlined should converge to an eigenvalue c .

The results obtained with this numerical procedure will be presented in a subsequent section.

APPROXIMATE ANALYTIC SOLUTIONS

The region in the neighborhood of $R = 0$ is of particular interest to the problem of wave formation in the melt layer of a body entering the atmosphere since the melt layer is usually highly viscous. Solutions valid in the neighborhood of $R = 0$ may be obtained by an expansion in powers of R as suggested by Yih (ref. 8). However, a somewhat simpler procedure than that employed by Yih can be used to obtain solutions for "fast" waves, that is, $\text{Re}(c) \gg 1$ where Re denotes real part. This procedure consists in neglecting the variable term η in comparison to the term $c - 1$ in the differential equation (41), solving the resulting equation, formulating the dispersion relation in analytic form valid for all values of R , and examining the behavior of the dispersion relation in the neighborhood of $R = 0$. This examination will be carried out by expanding the dispersion relation in a Taylor series in R . The expansion will be carried out to first order in R . This procedure should also yield approximate solutions when $|c - 1| \gg 1$. When the term η is omitted from equation (41), there are obtained the following differential equations with constant coefficients

$$\begin{aligned}\phi'' &= \alpha^2\phi + s \\ s'' &= \alpha^2s - i\alpha R(c - 1)s\end{aligned}$$

or

$$\left. \begin{aligned}\phi'' - \alpha^2\phi &= s \\ s'' - \beta^2s &= 0\end{aligned} \right\} \quad (52)$$

where

$$\beta^2 = \alpha^2 - i\alpha R(c - 1) \quad (53)$$

the boundary conditions (46) in terms of β are as follows:

$$\left. \begin{aligned}X\phi + (\beta^2 + \alpha^2)\phi' - s' &= 0 \\ s + 2\alpha^2\phi &= 0\end{aligned} \right\} \quad (54)$$

where

$$X = \frac{i\alpha}{c - 1} f - i\alpha R \quad (55)$$

The solutions of equations (52) are

$$\left. \begin{aligned}\phi &= Ae^{\alpha\eta} + Be^{-\alpha\eta} + Ce^{\beta\eta} + De^{-\beta\eta} \\ s &= (\beta^2 - \alpha^2)Ce^{\beta\eta} + (\beta^2 - \alpha^2)De^{-\beta\eta}\end{aligned} \right\} \quad (56)$$

where A , B , C , and D are arbitrary constants and where β denotes for definiteness the root with positive real part of equation (53). For a non-trivial solution of the eigenvalue problem, the substitution of equations (56) into equations (43) and (54) requires the vanishing of the following secular determinant:

$$\begin{vmatrix} e^{-\alpha} & e^{\alpha} & e^{-\beta} & e^{\beta} \\ \alpha e^{-\alpha} & -\alpha e^{\alpha} & \beta e^{-\beta} & -\beta e^{\beta} \\ X + \alpha(\beta^2 + \alpha^2) & X - \alpha(\beta^2 + \alpha^2) & X + 2\alpha^2\beta & X - 2\alpha^2\beta \\ 2\alpha^2 & 2\alpha^2 & \alpha^2 + \beta^2 & \alpha^2 + \beta^2 \end{vmatrix} = 0 \quad (57)$$

After some rearrangement the secular determinant may be written as follows:

$$\begin{vmatrix} \sinh \alpha & \sinh \beta & \cosh \alpha & \cosh \beta \\ \alpha \cosh \alpha & \beta \cosh \beta & \alpha \sinh \alpha & \beta \sinh \beta \\ \beta^2 + \alpha^2 & 2\alpha\beta & -X/\alpha & -X/\alpha \\ 0 & 0 & 2\alpha^2 & \alpha^2 + \beta^2 \end{vmatrix} = 0 \quad (57)$$

The dispersion relation for $R = 0$ may be obtained by evaluating the secular determinant equation (57) for $\beta = \alpha$ after cancelling the factor $(\beta - \alpha)^2$ from the determinant, and setting $\beta = \alpha$. The removal of $(\beta - \alpha)^2$ is required because the secular determinant is satisfied trivially for $\beta = \alpha$. To remove $(\beta - \alpha)^2$ the determinant (57) is rearranged as follows:

$$\begin{vmatrix} \sinh \alpha & \sinh \beta - \sinh \alpha & \cosh \alpha & \cosh \beta - \cosh \alpha \\ \alpha \cosh \alpha & \beta \cosh \beta - \alpha \cosh \alpha & \alpha \sinh \alpha & \beta \sinh \beta - \alpha \sinh \alpha \\ \beta^2 + \alpha^2 & -(\beta - \alpha)^2 & -X/\alpha & 0 \\ 0 & 0 & 2\alpha^2 & \beta^2 - \alpha^2 \end{vmatrix} = 0 \quad (58)$$

Substituting the identities

$$\sinh \beta = \sinh \alpha + 2 \cosh \frac{\beta + \alpha}{2} \sinh \frac{\beta - \alpha}{2}$$

$$\cosh \beta = \cosh \alpha + 2 \sinh \frac{\beta + \alpha}{2} \sinh \frac{\beta - \alpha}{2}$$

into the determinant and dividing the second and fourth columns of the resulting determinant by the factor $(\beta - \alpha)$ yields

$$\begin{vmatrix} \sinh \alpha & \frac{2 \cosh \frac{\beta + \alpha}{2} \sinh \frac{\beta - \alpha}{2}}{\beta - \alpha} & \cosh \alpha & \frac{2 \sinh \frac{\beta + \alpha}{2} \sinh \frac{\beta - \alpha}{2}}{\beta - \alpha} \\ \alpha \cosh \alpha & \frac{2\beta \sinh \frac{\beta + \alpha}{2} \sinh \frac{\beta - \alpha}{2} + \cosh \alpha}{\beta - \alpha} & \alpha \sinh \alpha & \frac{2\beta \cosh \frac{\beta + \alpha}{2} \sinh \frac{\beta - \alpha}{2} + \sinh \alpha}{\beta - \alpha} \\ \beta^2 + \alpha^2 & -(\beta - \alpha) & \frac{-X}{\alpha} & 0 \\ 0 & 0 & 2\alpha^2 & \beta + \alpha \end{vmatrix} = 0 \quad (59)$$

letting $\beta = \alpha$ (noting that at $\beta = \alpha$, $2 \sinh[(\beta - \alpha)/2]/(\beta - \alpha) = 1$) yields

$$\begin{vmatrix} \sinh \alpha & \cosh \alpha & \cosh \alpha & \sinh \alpha \\ \alpha \cosh \alpha & \alpha \sinh \alpha + \cosh \alpha & \alpha \sinh \alpha & \alpha \cosh \alpha + \sinh \alpha \\ 2\alpha^2 & 0 & -X/\alpha & 0 \\ 0 & 0 & 2\alpha^2 & 2\alpha \end{vmatrix} = 0 \quad (60)$$

Evaluation of the determinant yields

$$(\sinh \alpha \cosh \alpha - \alpha)(-X/\alpha) = 2\alpha^2(\alpha^2 + \cosh^2 \alpha) \quad (61)$$

With X given by equation (55), equation (61) yields at $R = 0$ for the subsonic case

$$c - 1 = \frac{i}{2\alpha} \frac{1}{c_f \sqrt{1 - M^2}} \frac{\sinh \alpha \cosh \alpha - \alpha}{\alpha^2 + \cosh^2 \alpha} \quad (62)$$

and for the supersonic case,

$$c - 1 = \frac{1}{2\alpha} \frac{1}{c_f \sqrt{M^2 - 1}} \frac{\sinh \alpha \cosh \alpha - \alpha}{\alpha^2 + \cosh^2 \alpha} \quad (63)$$

It should be noted that equations (62) and (63) are valid at $R = 0$ regardless of the value of c . This can be seen by carrying out Yih's expansion procedure for obtaining the dispersion relations. To first order ($R = 0$), the coefficient of R in the differential equation (41) does not contribute to the solution, so that neglecting the term η in comparison to the term $c - 1$ is a superfluous assumption for $R = 0$. However, this is not the case in the neighborhood of $R = 0$.

It will be shown subsequently that the dispersion relation in analytic form (eq. (59)), is extremely useful especially when it is complemented with the numerical method previously presented.

The physical interpretation of the limiting case $R = 0$ is discussed by Yih (ref. 8) in connection with the stability of liquid flow down an inclined plane. In the case $R = 0$ there is no disturbance motion, but merely a surface corrugation that has no reason to be damped. Since the quantity c is expressed in terms of a reference velocity, U_1 , the actual dimensional phase velocity and amplification factor in the case of a surface corrugation would be zero since $U_1 = 0$ for a liquid layer with a large but finite viscosity.

Of considerable significance is the behavior of the dispersion relation in this case of static corrugation. Equation (62) indicates that the subsonic solutions are unstable at $R = 0$. This apparently surprising result can be

clarified by realizing that the destabilizing action of the gas pressure as indicated by equation (23) assumes a dominating role near $R = 0$ (see eq. (38)). Equation (63) indicates that the line $R = 0$ is a neutral curve for the supersonic case. The behavior of c in the neighborhood of $R = 0$ can be determined from equations (53) and (59). Differentiation of the terms in equation (53) regarding R as the independent variable yields at $R = 0$

$$\frac{dc}{dR} = \frac{dc}{d\beta} \frac{d\beta}{dR} = -i \frac{(c - 1)}{2} \frac{dc}{d\beta} \quad (64)$$

Now equation (59) can be regarded as an implicit equation connecting c and β , and the derivative $dc/d\beta$ can be evaluated at $R = 0$ by differentiating the determinant (59) with respect to β . This calculation is displayed in appendix B. The result for the subsonic case is that the solutions become less unstable as R increases from zero, indicating that all terms in equation (38) come into play. A further result is that the real part of c decreases as R increases from zero, indicating, since $\text{Re}(c) = 1$ at $R = 0$, that the subsonic case does not exhibit the type of instability being examined (phase velocity $\text{Re}(c)$ greater than unity). Hence, the subsonic case will no longer be considered; instead, attention will be devoted to the supersonic case only. At $R = 0$ the result for the supersonic case is as follows:

$$\frac{dc}{dR} = i \left\{ \frac{(c')^2}{4\alpha} \left[3 - \frac{2\alpha^2 \sinh \alpha (\cosh \alpha + \alpha \sinh \alpha)}{(\alpha^2 + \cosh^2 \alpha) (\sinh \alpha \cosh \alpha - \alpha)} \right] - \frac{c'}{\alpha} c_f \sqrt{M^2 - 1} \left[\frac{1}{F^2} + \frac{\alpha^2}{W^2} - c' \right] \right\} \quad (65)$$

where $c' = c - 1$ and c is given by equation (63).

Now

$$c(R) = c(0) + \left(\frac{dc}{dR} \right)_{R=0} R + \dots \quad (66)$$

and since the right-hand side of equation (65) is purely imaginary, this equation gives the damping or amplification of the disturbance in the neighborhood of $R = 0$. Setting $dc/dR = 0$ in equation (65) yields a relation among the parameters on the right-hand side valid for neutral disturbances. These relations will be presented subsequently, after the approximation inherent in equation (65) is evaluated by comparing the results of this equation with the numerical results. Before presenting these further results, it is of interest to examine solutions of the eigenvalue problem for long waves (small α).

For finite values of R the solution of the eigenvalue problem may be obtained in the neighborhood of $\alpha = 0$ by expansion in powers of α as suggested by Yih, reference 8. Solutions of equations (41) are sought near $\alpha = 0$ with c near 1 (eq. (52) indicates that $c = 1$ at $\alpha = 0$). The procedure set forth for carrying out the numerical solution will be used for this analytic solution. The initial conditions to be used are equations (46).

Let

$$x = x_0 + \alpha x_1 + \dots$$

$$c = 1 + \alpha c_1 + \dots$$

$$\phi = \phi_0 + \alpha \phi_1 + \dots$$

$$s = s_0 + \alpha s_1 + \dots$$

Substituting these relations in equation (41) yields to lowest order in α

$$\left. \begin{aligned} \phi_0'' &= s_0 \\ s_0'' &= 0 \end{aligned} \right\} \quad (67)$$

the initial conditions equations (46) to lowest order in α are at $\eta = 0$

$$\left. \begin{aligned} \phi &= 1 \\ \phi' &= x_0 \\ s &= 0 \\ s' &= \frac{iR}{c_1} \frac{1}{F^2} \end{aligned} \right\} \quad (68)$$

The satisfaction of the boundary conditions at $\eta = -1$: equations (43) yield two equations for the determination of the two quantities x_0 and c_1 . This yields the following dispersion relation near $\alpha = 0$

$$c = 1 - \frac{i\alpha}{3} \frac{R}{F^2} \quad (69)$$

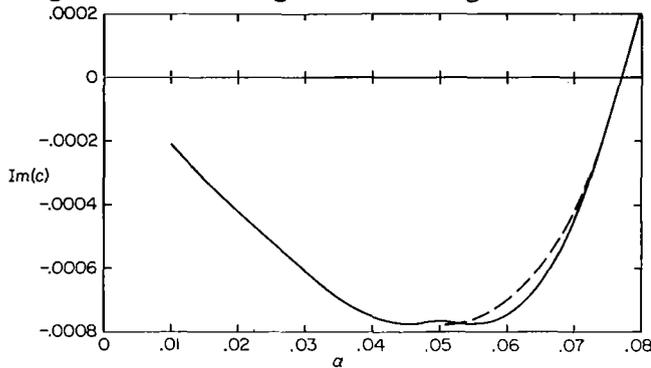
This expression indicates that the line $\alpha = 0$ is a neutral curve. Furthermore, it can be verified that dc/dR computed from equation (69) agrees with equation (65) to lowest order in α .

To summarize the results of this section for the supersonic case, we can say that the line $\alpha = 0$ is a neutral curve, that the line $R = 0$ is a neutral curve, that the rate of amplification near the line $R = 0$ may be obtained, approximately, from equation (65) and that the relation between the parameters for neutral disturbances near the line $R = 0$ may be obtained, approximately, from equation (65) by setting the left-hand side equal to zero. The validity of the approximation inherent in equation (65) will be assessed for selected values of the parameters in the following section.

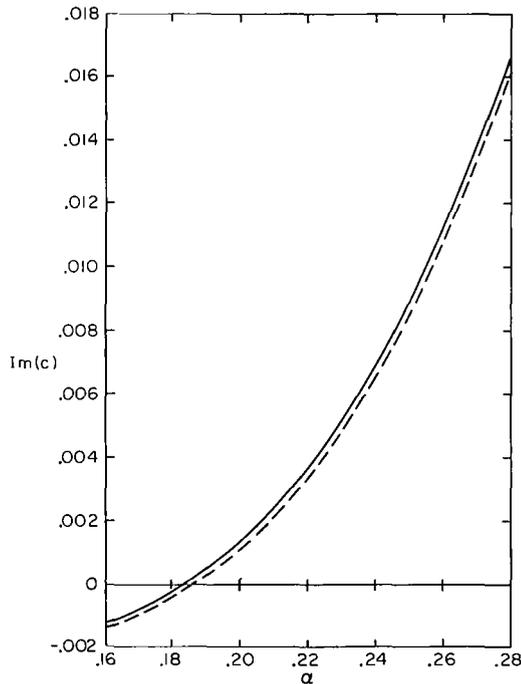
Comparison of Approximate Solutions With Numerical Solutions

The numerical procedure described previously was programmed on the IBM 7094 digital computer at Ames Research Center. Single precision complex arithmetic was used. The numerical integration was started with the Runge-Kutta formulas and continued with the Adams-Moulton formulas.

The step size was externally controlled and set so that there was no significant change in the eigenvalue when the example was rerun with a step



(a) $c_F \sqrt{M^2 - 1} = 0.001$



(b) $c_F \sqrt{M^2 - 1} = 0.005$

Figure 2.- Comparison of numerical (dashed line) and approximate (solid line) determination of a neutral point at $R = 0.01$ for same value of parameters as given in figure 4.

size equal to half of the original value. The step size of $1/32$ was found to be satisfactory for most runs. To assure convergence to an eigenvalue the sum of the absolute values of the discrepancies of the computed boundary values was required to be less than 5×10^{-5} .

The approximate analytical solutions obtained in the previous section were also obtained by means of the numerical procedure for selected values of the pertinent parameters.

Because of the large number of parameters required to describe the physical configuration under study, and the ranges of these parameters, analytic formulas, such as equations (63) and (65), are helpful for describing the results of the stability analysis. Although they may be approximate, analytic solutions also simplify the searching process usually required to find a neutral point.

The degree of approximation inherent in equation (65) may be determined from a comparison of the evaluations obtained from that equation with the numerical solutions. These solutions are compared in figure 2 which gives the numerical solutions for $R = 0.01$ and values of $\text{Im}(c)$ (Im denotes imaginary part). Also shown in figure 2 are the extrapolations obtained from equation (66) with $R = 0.01$. It can be seen from this figure that the approximate solution provides values close to those obtained by the numerical method. For example,

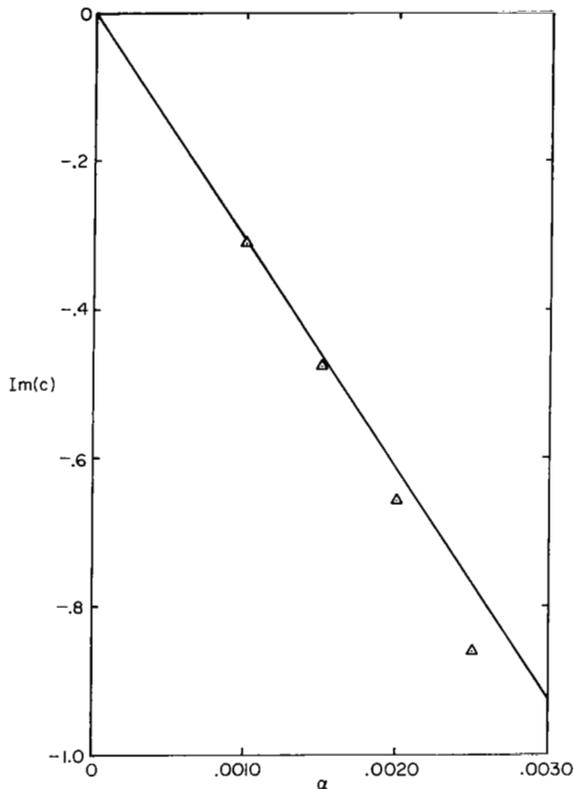


Figure 3.- Comparison of numerical solutions marked (Δ) with equation (69) shown as a straight line at $R = 150$; $c_f/\sqrt{M^2 - 1} = 0.001$, $U_1 = 4$ cm/sec, $h = 0.1$ cm, $\bar{T} = 72$ dynes/cm, $g = 980$ cm/sec².

it can be seen from figure 2(b) that the numerical solutions yield a value of $\alpha = 0.187$; whereas the approximate solutions yield $\alpha = 0.185$ for a neutral disturbance ($\text{Im}(c) = 0$). This agreement is considered sufficiently accurate for the stability results. The parameters used to describe the kinetics of the liquid film are the depth of the film, h , and the velocity of the film at the interface, U_1 . The results shown in figure 2 apply to a film with $h = 0.1$ cm and $U_1 = 4$ cm/sec.

The numerical solutions can be checked by means of equation (69). The results obtained from equation (69) are exact for vanishingly small α and are compared with numerical results at $R = 150$ in figure 3.

These two comparisons for selected values of the parameters lend credence to both the numerical and approximate analytic results. The results obtained for a more extensive range of parameters are presented in the next section.

RESULTS AND DISCUSSION

The results of a stability analysis are usually displayed in an $\alpha - R$ diagram on which the neutral curve ($\text{Im}(c) = 0$) is drawn. As previously stated, the line $\alpha = 0$ is a neutral curve and for the supersonic case, the line $R = 0$ is a neutral curve. However, there are also neutral curves with α different from zero emanating from the line $R = 0$, for example, see figure 2. Starting from these points near $R = 0$, the neutral curves were traced as R increased by solving the eigenvalue problem numerically. The results of these computations are shown in figure 4.

An interesting question concerning the neutral curves in figure 4 is why does one pass from region of instability to stability as the neutral curve is crossed when R is increased at a fixed value of α . This result is contrary to the usual situation (Tollmien-Schlichting instabilities). The answer to the question lies in clarifying the source of energy for the two types of disturbance motion. In the Tollmien-Schlichting instability with $c < 1$ the disturbance energy is supplied by the mean motion of the film. In the

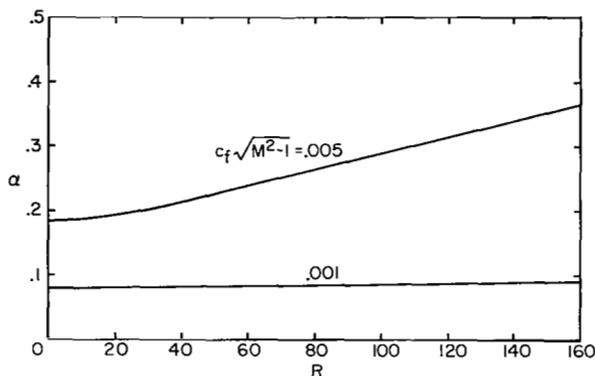


Figure 4.- Neutral curve for various values of the parameter $c_f\sqrt{M^2-1}$. Region below each of the curves is stable. $U_1 = 4$ cm/sec, $h = 0.1$ cm, $T = 72$ dynes/cm, $g = 980$ cm/sec².

present case with $c > 1$ the disturbance energy of the film is supplied by the motion of the gas.

From equation (37) it can be seen that as the Reynolds number R of the film is increased for fixed c_f the influence of the last term in the bracket on the right-hand side of that equation is reduced. The other two terms in the bracket always tend to stabilize the disturbance motion for the cases shown in figure 4. These two terms become dominant when R is sufficiently large.

The last term in the bracket on the right-hand side of equation (37)

is proportional to the pressure coefficient for supersonic flow past a wavy wall (ref. 9). Furthermore, the disturbance pressure is not in phase with the wall and a wave drag is exerted on the wall (the disturbed gas-liquid interface in the present case). These phase relations are analogous in a sense to the phase relations in the Tollmien-Schlichting type instability where the mechanism for wave generation is the change in phase across the liquid layer (ref. 10). However, as opposed to the Tollmien-Schlichting type of instability where a description of the mechanism is rather involved, the mechanism of wave generation in the present case can simply be described as supersonic wave drag. Because the oblique waves under consideration are three dimensional, it is not only necessary that the external stream be supersonic, but also that the component of the external flow across the crests of the waves be supersonic.

Since these general properties are evident in the $\alpha - R$ diagram for a few selected values of the parameters, this representation will not be pursued further as a means for displaying the results of the analysis. Rather, another representation will be used which is better suited for displaying the results for small values of R .

Because equation (65) provides sufficiently accurate answers for small R , the case of interest, it will form the basis for presenting the results for neutral disturbances covering a wide range of parameters. Also, it should be noted from figure 4 that the neutral curves are nearly horizontal, so that the values of α established for neutral points by means of equation (65) near $R = 0$ for a given value of $c_f\sqrt{M^2-1}$ should correspond approximately to the same value of $c_f\sqrt{M^2-1}$ at much larger values of R .

The values of α corresponding to neutral points may be established if one plots equation (65) as a function of α for given values of the parameters and notes where the curve crosses zero. However, it is more convenient

to set the left-hand side of equation (65) to zero and solve the resulting equation. This solution is accomplished by eliminating the term $c_f \sqrt{M^2 - 1}$ by means of equation (63) and rearranging to obtain the following quadratic equation in c'

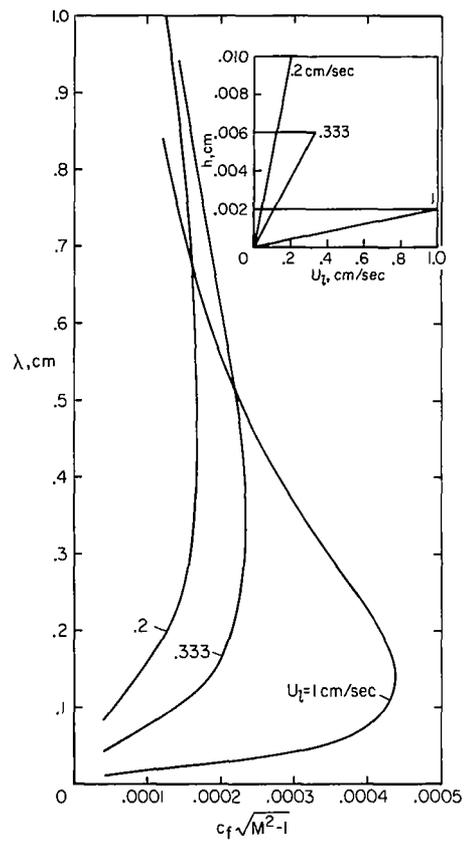
$$\frac{\alpha}{2(\sinh \alpha \cosh \alpha - \alpha)} \left(3 \cosh^2 \alpha + \alpha^2 - \frac{2\alpha^3 \cosh^2 \alpha}{\sinh \alpha \cosh \alpha - \alpha} \right) c'^2 + c' - \left(\frac{1}{F^2} + \frac{\alpha^2}{W^2} \right) = 0$$

(70)

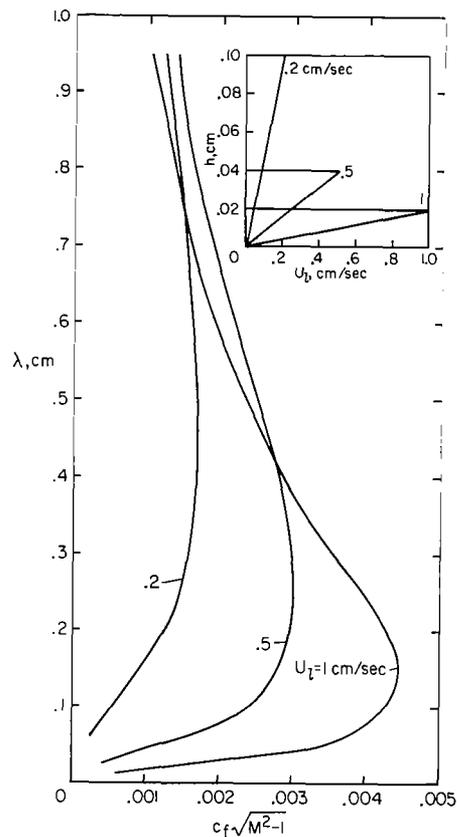
As stated previously the kinematics of the liquid film are characterized by h and U_1 . Given these values and values of α , the surface tension coefficient T , and the acceleration of gravity g , all the coefficients in equation (70) can be evaluated and the resulting quadratic equation can be solved for c' . The choice of the sign before the radical in the quadratic formula can be decided by noting that for large α the coefficient of c'^2 is positive. This fact requires that the positive sign be chosen for positive roots (c' positive). After the quadratic equation is solved the parameter $c_f \sqrt{M^2 - 1}$ is evaluated by means of equation (63). This leads to

$$c_f \sqrt{M^2 - 1} = \frac{\sinh \alpha \cosh \alpha - \alpha}{2\alpha c' (\alpha^2 + \cosh^2 \alpha)}$$

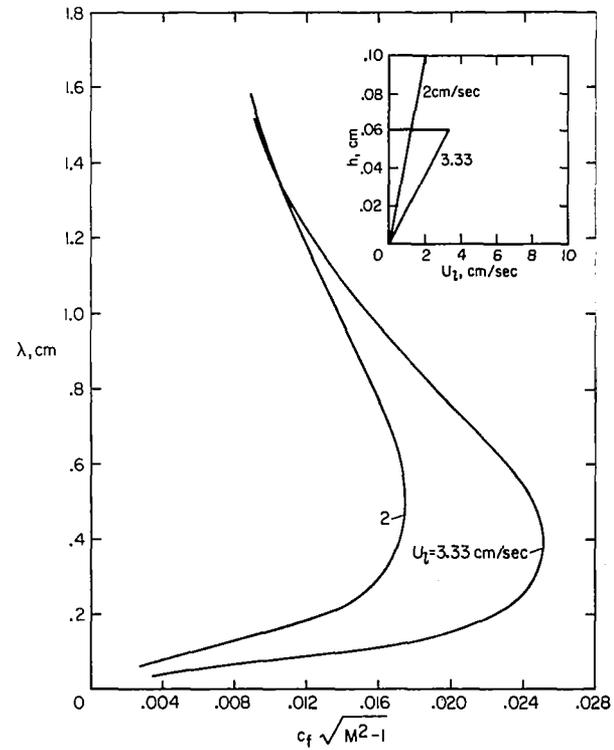
The results for neutral disturbances are displayed in dimensional form by plotting the wavelength of the disturbance $\lambda = 2\pi h/\alpha$ against $c_f \sqrt{M^2 - 1}$. These results are displayed in figure 5. On each figure the quantity hU_1 is held constant. The quantity $(1/2)hU_1 \equiv Q'$ is the volume flow rate per unit width in the direction of the wave normal. Note that for a given value of $c_f \sqrt{M^2 - 1}$, there are either two neutral wavelengths or none. It should be noted that the Reynolds number of the film R is not specified for any of the results in figure 5. As stated previously these results should hold over a range of Reynolds numbers and become increasingly more accurate as the Reynolds number approaches zero. For wavelengths much smaller than the depth of the film the results in figure 5 may be compared with the results of Willson and Chang (ref. 11) who treated the case of an infinitely deep liquid. One of their results is that the wavelength is an increasing function of the Mach number. This trend is evident from an inspection of figure 5 for short wavelengths. However, in contrast to their result, the wavelength does not approach an asymptotic value as the Mach number increases. One should be reminded that the variables and parameters used in figure 5 correspond to quantities measured in the direction of the wave normal, not in the direction of the external stream.



(a) $Q' = 0.001 \text{ cm}^2/\text{sec}$



(b) $Q' = 0.01 \text{ cm}^2/\text{sec}$



(c) $Q' = 0.1 \text{ cm}^2/\text{sec}$

Figure 5.- Film wavelength versus stream parameters for various depths of film; $T = 21 \text{ dynes/cm}$, $g = 980 \text{ cm/sec}^2$. Region to the right of each curve is stable.

CONCLUSIONS

It is concluded from the foregoing linear analysis, that oblique waves will be generated in the melt layer of a body entering the atmosphere even though the layer is extremely viscous. The mechanism of the wave generation is supersonic wave drag. For the type of oblique waves considered, the direction of the wave crests is inclined not at the Mach angle of the external stream but the direction is such so that there is always supersonic flow across their crests. The wavelength of neutral disturbances is governed by parameters associated with the external stream and with the melt layer itself.

Ames Research Center

National Aeronautics and Space Administration

Moffett Field, Calif., 94035, Aug. 28, 1968

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APPENDIX A

PRINCIPAL NOMENCLATURE

a	speed of sound
c	phase velocity, dimensionless
f	defined by equation (44) or (45)
F'	velocity of liquid
h	depth of film
i	imaginary unit
Im()	imaginary part
k	wave number
p	pressure
P	pressure, basic flow
Q'	$(1/2)U_1h$ quantity of flow per unit width
Re()	real part
s	disturbance vorticity, dimensionless
t	time
T	surface tension coefficient
u	velocity component
U	velocity component, basic flow
x	$\phi'(0)$
X	defined by equation (55)
x_1, x_2, x_3	Cartesian coordinates
α	wave number, dimensionless
β	defined by equation (53)
γ	ratio of specific heats

ϵ	amplitude of disturbance of the interface
η	coordinate normal to surface, dimensionless
λ	wavelength
μ	dynamic viscosity, liquid
ν	kinematic viscosity, liquid
ρ	density
τ	stress components, gas
ϕ	disturbance stream function, dimensionless
ω	angular frequency of disturbance

Subscripts

c	differentiation with respect to c
g	gas quantity evaluated at gas-liquid interface
l	liquid quantity evaluated at gas-liquid interface
x	external flow direction and differentiation with respect to x
α, β	tensor index, range (1-2)
∞	external stream conditions

Superscript

$(\bar{\quad})$	total quantity
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Dimensionless Groups

c_f	friction coefficient
F	Froude number
M	Mach number
R	Reynolds number
W	Weber number

APPENDIX B

CALCULATION OF DERIVATIVE

Differentiation of the determinant (eq. (59)) with respect to β and evaluation at $\beta = \alpha$ yields

$$\begin{aligned}
 & \begin{vmatrix} 0 & \cosh \alpha & \cosh \alpha & \sinh \alpha \\ 0 & \alpha \sinh \alpha + \cosh \alpha & \alpha \sinh \alpha & \alpha \cosh \alpha + \sinh \alpha \\ 2\alpha & 0 & \frac{-X}{\alpha} & 0 \\ 0 & 0 & 2\alpha^2 & 2\alpha \end{vmatrix} \\
 + \frac{1}{2} & \begin{vmatrix} \sinh \alpha & \sinh \alpha & \cosh \alpha & \sinh \alpha \\ \alpha \cosh \alpha & \alpha \cosh \alpha + 2 \sinh \alpha & \alpha \sinh \alpha & \alpha \cosh \alpha + \sinh \alpha \\ 2\alpha^2 & -2 & \frac{-X}{\alpha} & 0 \\ 0 & 0 & 2\alpha^2 & 2\alpha \end{vmatrix} \\
 + & \begin{vmatrix} \sinh \alpha & \cosh \alpha & 0 & \sinh \alpha \\ \alpha \cosh \alpha & \alpha \sinh \alpha + \cosh \alpha & 0 & \alpha \cosh \alpha + \sinh \alpha \\ 2\alpha^2 & 0 & \frac{d}{d\beta} \left(\frac{-X}{\alpha} \right) & 0 \\ 0 & 0 & 0 & 2\alpha \end{vmatrix} \\
 + \frac{1}{2} & \begin{vmatrix} \sinh \alpha & \cosh \alpha & \cosh \alpha & \cosh \alpha \\ \alpha \cosh \alpha & \alpha \sinh \alpha + \cosh \alpha & \alpha \sinh \alpha & \alpha \sinh \alpha + 2 \cosh \alpha \\ 2\alpha^2 & 0 & \frac{-X}{\alpha} & 0 \\ 0 & 0 & 2\alpha^2 & 2 \end{vmatrix} = 0
 \end{aligned}$$

Evaluation of the four determinants in order yields

$$\begin{aligned}
& -4\alpha^4 - 4\alpha^2 \cosh^2 \alpha + 2\alpha \sinh^2 \alpha \left(\frac{-X}{\alpha} \right) \\
& -4\alpha^3 \sinh \alpha \cosh \alpha - 2\alpha^2(1 + \alpha^2) \cosh^2 \alpha + 4\alpha^4 \sinh^2 \alpha \\
& + 2\alpha(\sinh \alpha \cosh \alpha - \alpha) \frac{d}{d\beta} \left(\frac{-X}{\alpha} \right) \\
& \quad + (\sinh \alpha \cosh \alpha - \alpha) \left(\frac{-X}{\alpha} \right) - 2\alpha^2(1 + \alpha^2) \cosh^2 \alpha = 0
\end{aligned}$$

Combining like terms yields

$$\begin{aligned}
& -4\alpha^2(2\alpha^2 + 2 \cosh^2 \alpha + \alpha \sinh \alpha \cosh \alpha) \\
& + \left(\frac{-X}{\alpha} \right) (2\alpha \sinh^2 \alpha + \sinh \alpha \cosh \alpha - \alpha) \\
& \quad + 2\alpha \frac{d}{d\beta} \left(\frac{-X}{\alpha} \right) (\sinh \alpha \cosh \alpha - \alpha) = 0 \quad (B1)
\end{aligned}$$

SUPERSONIC CASE

Evaluating equation (55) at $R = 0$ yields

$$X = \frac{i\alpha}{c - 1} f$$

From equation (44) evaluated at $R = 0$

$$f = \frac{i\alpha}{c_f} \frac{1}{\sqrt{M^2 - 1}}$$

Hence,

$$\left(\frac{-X}{\alpha} \right) = \frac{\alpha}{c - 1} \frac{1}{c_f \sqrt{M^2 - 1}} = \frac{2\alpha^2(\alpha^2 + \cosh^2 \alpha)}{\sinh \alpha \cosh \alpha - \alpha} \quad (B2)$$

where the second equality follows after equation (63) is used.

In order to evaluate $(d/d\beta)X$ it is convenient to express X in terms of β rather than R . From equations (44), (53), and (55) we have

$$-X = (\beta^2 - \alpha^2)(c - 1)^{-2} \left(\frac{1}{F^2} + \frac{\alpha^2}{W^2} \right) + \frac{\alpha^2(c - 1)^{-1}}{c_f \sqrt{M^2 - 1}} - (\beta^2 - \alpha^2)(c - 1)^{-1}$$

Differentiation of this expression with respect to β taking into account that c is to be regarded as a function of β yields

$$\frac{1}{\alpha} \left[\frac{d}{d\beta} (-X) \right]_{\beta=\alpha} = 2 \left[\frac{1}{(c-1)^2} \left(\frac{1}{F^2} + \frac{\alpha^2}{W^2} \right) - \frac{1}{c-1} \right] - \frac{\alpha}{(c-1)^2} \frac{1}{c_f \sqrt{M^2-1}} \frac{dc}{d\beta} \quad (B3)$$

Substituting equations (B2) and (B3) into equation (B1) and rearranging yields

$$- \frac{(c-1)}{\alpha} \frac{dc}{d\beta} = \frac{(c')^2}{4\alpha} \left[3 - \frac{2\alpha^2 \sinh \alpha (\cosh \alpha + \alpha \sinh \alpha)}{(\alpha^2 + \cosh^2 \alpha) (\sinh \alpha \cosh \alpha - \alpha)} \right] - \frac{c'}{\alpha} c_f \sqrt{M^2-1} \left(\frac{1}{F^2} + \frac{\alpha^2}{W^2} - c' \right)$$

This is the desired result, since

$$\frac{dc}{dR} = -i \frac{(c-1)}{2} \frac{dc}{d\beta}$$

SUBSONIC CASE

Substituting equation (45) into equation (55) and evaluating at $R = 0$ yields

$$\frac{-X}{\alpha} = \frac{-i\alpha}{c-1} \frac{1}{c_f \sqrt{1-M^2}} = \frac{-2\alpha^2 (\alpha^2 + \cosh^2 \alpha)}{\sinh \alpha \cosh \alpha - \alpha} \quad (B4)$$

where the second equality follows after equation (62) is used.

For the evaluation of the derivative, X is expressed in terms of β as follows:

$$-X = (\beta^2 - \alpha^2) (c-1)^{-2} \left(\frac{1}{F^2} + \frac{\alpha^2}{W^2} \right) + \frac{i\alpha^2}{c_f \sqrt{1-M^2}} (c-1)^{-1} - (\beta^2 - \alpha^2) (c-1)^{-1}$$

Differentiating this expression with respect to β , regarding c as a function of β as before, yields:

$$\frac{1}{\alpha} \left[\frac{d}{d\beta} \left(\frac{-X}{\alpha} \right) \right]_{\beta=\alpha} = 2 \left[\frac{1}{(c-1)^2} \left(\frac{1}{F^2} + \frac{\alpha^2}{W^2} \right) - \frac{1}{c-1} \right] - \frac{i\alpha}{c_f \sqrt{1-M^2}} \frac{1}{(c-1)^2} \frac{dc}{d\beta} \quad (B5)$$

Substituting equations (B4) and (B5) into equation (B1) and rearranging yields

$$\begin{aligned}
 \frac{dc}{dR} &= -i \frac{(c-1)}{2} \frac{dc}{d\beta} \\
 &= -\frac{1}{4\alpha^3} \frac{1}{c_f \sqrt{1-M^2}} \left(\frac{\sinh \alpha \cosh \alpha - \alpha}{\alpha^2 + \cosh^2 \alpha} \right)^2 - \frac{i}{2\alpha^2} \frac{\sinh \alpha \cosh \alpha - \alpha}{\alpha^2 + \cosh^2 \alpha} \left(\frac{1}{F^2} + \frac{\alpha^2}{W^2} \right) \\
 &\quad + i \left[\frac{(c-1)^2}{4\alpha} \left(\frac{4\alpha^2 + 4 \cosh^2 \alpha + 2\alpha \sinh \alpha \cosh \alpha}{\alpha^2 + \cosh^2 \alpha} \right. \right. \\
 &\quad \left. \left. + \frac{2\alpha \sinh^2 \alpha + \sinh \alpha \cosh \alpha - \alpha}{\sinh \alpha \cosh \alpha - \alpha} \right) \right]
 \end{aligned}$$

The real part of this expression indicates that $\text{Re}(dc/dR)$ is negative at $R = 0$. The imaginary part indicates that $\text{Im}(dc/dR)$ is also negative at $R = 0$.

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